GRADE 9 | UNIT 5

Rational Exponents and Radical Expressions

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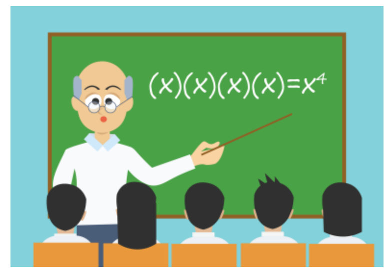
UNIT 5

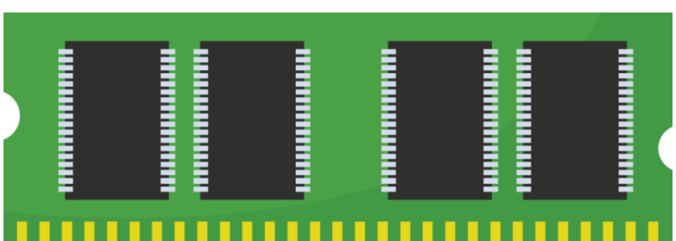
Rational Exponents and Radical Expressions



Your first experience of exponents could be a number multiplied by itself many times or a polynomial where a variable is raised to a second or third degree. Similarly, you are acquainted with the square root or cube root of a number.

In this unit, we will explore more about rational exponents and radical expressions and how essential they are in real life applications.

Rational exponents are useful to financial analysts to predict the rate of inflation or to calculate the compound interest of money over a period of time. One can readily check the amount of amortization for a certain amount of loan or even the interest of bank deposit for a chosen period of time.



Computer experts must have a deep understanding of exponents to work efficiently with computers. Computer file size is measured using megabytes, which is equivalent to 10^6 bytes. The hard drive capacity is usually measured in gigabytes (10^9) and terabytes (10^{12}). Computer memory, specifically the Random Access Memory (RAM), is also measured in gigabytes and the speed of the processor is measured in gigahertz.

Let us discover deeper the concept of exponents and radicals in this unit.

Test Your Prerequisite Skills

- Expressing numbers in exponential form
- Multiplying integers and fractions
- Adding and subtracting fractions
- Multiplying binomials

Before you get started, answer the following items on a separate sheet of paper. This will help you assess your prior knowledge and practice some skills that you will need in studying the lessons in this unit. Show your complete solution.

1. Express the following numbers in exponential form using prime factors.
   a. 64
   b. 49
   c. 81
   d. 225
   e. 343

2. Multiply the following integers.
   a. $(-13)(12)$
   b. $(-9)(-16)$
   c. $\frac{2}{3}(-24)$

3. Perform the indicated operation. Simplify the answer if necessary.
   a. $\frac{3}{4}+\left(-\frac{1}{2}\right)$
   b. $-\frac{2}{5}-\left(-\frac{5}{6}\right)$

4. Multiply the following binomials.
   a. $(2x+3)(x+1)$
   b. $(3xy-1)(xy+3)$

Objectives

At the end of this unit, you should be able to

- apply the laws involving positive integral exponents to zero and negative integral exponents;
- illustrate expressions with rational exponents;
- write expressions with rational exponents as radicals and vice versa;
- simplify expressions with rational exponents;
- derive the laws of radicals;
- simplify radical expressions using the laws of radicals;
- perform operations on radical expressions;
- solve equations involving radical expressions; and
- solve problems involving radicals.

Lesson 1: Zero and Negative Integral Exponents

Warm Up!

Bunnies in Twos!

Materials Needed: pen and paper

Instructions:

1. This activity is done individually.
2. Using the following chart, complete the boxes with the correct information as described.

Rabbit is one of the fastest growing populations in the animal kingdom. Their population roughly triples every three months. If a certain habitat starts with two rabbits, can you determine the number of rabbits in the given time?

Habitat A				
Number of Rabbits	After 3 months	After 6 months	After 9 months	After 12 months
2	2×3	$2 \times 3 \times 3$		
$2^1 = 2$	$2 \times 3 = 6$	$2 \times 3^2 = 18$		

Habitat B				
Number of Rabbits	After 3 months	After 6 months	After 9 months	After 12 months
8	8×3	$8 \times 3 \times 3$		
$8^1 = 2$	$8 \times 3 = 24$	$8 \times 3^2 = 72$		

3. Determine the total population of rabbits for both habitats after 18 months.

Learn about It!

In the *Warm Up!* activity, if you continue counting the rabbits and follow the exponential pattern, you can develop some rules about exponents.

At this point, we have mastered the application of **positive integral exponents**. Let us go through a quick review of exponential notation:

$$a^n = \underbrace{(a)(a)(a)(a)\cdots(a)}_{(n \text{ factors})}$$

In essence, exponent refers to the number of times a number is used as a factor.

Take the case of 2^4. The **base** is the number being multiplied. In this case, it is 2, and the **exponent** is 4.

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

What if the base is raised to a negative power?

We can dissect this by using the progression of 2^4 below:

$$\begin{aligned} 2^4 &= 2 \times 2 \times 2 \times 2 \\ 2^3 &= 2 \times 2 \times 2 \\ 2^2 &= 2 \times 2 \\ 2^1 &= 2 \\ 2^0 &= 1 \\ 2^{-1} &= \frac{1}{2} = \frac{1}{2^1} \\ 2^{-2} &= \frac{1}{4} = \frac{1}{2^2} \end{aligned}$$

If you notice the progression, you see that as the exponent decreases from 4 to -2, the number of 2's also decreases.

We can proceed to generalize this into the definition below:

For any non-zero real number b and integer n,

$$\boldsymbol{b^{-n} = \frac{1}{b^n}}$$

The same rules that apply to positive integral exponents are also applicable to negative integral exponents.

1. **Product Rule**: For any non-zero real number a and integers m and n,

$$a^{-m}a^{-n} = a^{-m-n}.$$

2. **Quotient Rule:** For any non-zero real number a and integer m and n,

$$\frac{a^{-m}}{a^{-n}} = a^{-m-(-n)} = a^{-m+n}.$$

3. **Power of a Power Rule:** For any non-zero real number a and integer m and n,

$$(a^{-m})^{-n} = a^{(-m)(-n)} = a^{mn}.$$

4. **Power of a Product Rule:** For any non-zero real numbers a and b and integers m, n, p,

$$(a^{-m}b^{-n})^{-p} = a^{(-m)(-p)} * b^{(-n)(-p)}.$$

5. **Power of a Quotient Rule:** For any non-zero real numbers a and b and integers m, n, p,

$$\left(\frac{a^{-m}}{b^{-n}}\right)^{-p} = \frac{a^{(-m)(-p)}}{b^{(-n)(-p)}}.$$

Moreover, any non-zero number raised to 0 is equal to 1. This statement may be verified using the Quotient Rule. Recall that $\frac{a^m}{a^n} = a^{m-n}$. We may remember as well that any non-zero number divided by itself is equal to 1. For example,

$$\frac{5}{5} = 1$$

We may rewrite the left-hand side of the equation as

$$\frac{5^1}{5^1} = 1$$

Let us utilize the Quotient Rule to again, rewrite the left-hand side of the equation.

$$5^{1-1} = 1$$
$$5^0 = 1$$

We can use the same process to generalize that any nonzero number raised to 0 is equal to 1.

Let's Practice!

Example 1: Simplify the expression $4^5 \times \frac{4^3}{4^6}$.

Solution: Apply the Product and the Quotient Rule.

Step 1: Apply the Quotient Rule to the expression $\frac{4^3}{4^6}$.

$$\frac{4^3}{4^6} = 4^{3-6}$$
$$\frac{4^3}{4^6} = 4^{3-6} = 4^{-3}$$

Step 2: Apply the Product Rule in the resulting expression.

$$4^5 \times 4^{-3} = 4^{5+(-3)}$$
$$4^5 \times 4^{-3} = 4^{5+(-3)} = 4^{5-3}$$
$$4^5 \times 4^{-3} = 4^2$$
$$4^5 \times 4^{-3} = 16$$

Thus, $4^5 \times \frac{4^3}{4^6} = 16$.

Try It Yourself!

Simplify the expression $3^4 \times \frac{3^5}{3^2}$.

Example 2: Simplify $(-2x^3y^{-4})^{-2}$.

Solution: Apply the Power of a Product Rule to simplify the expression.

Step 1: Raise each of the factors inside the parentheses to the exponent -2.

$$(-2x^3y^{-4})^{-2} = (-2)^{-2}(x^3)^{-2}(y^{-4})^{-2}$$

Step 2: Apply the Power of a Power Rule.

$$(-2x^3y^{-4})^{-2} = (-2)^{-2}(x^3)^{-2}(y^{-4})^{-2}$$
$$(-2x^3y^{-4})^{-2} = \frac{1}{(-2)^2}x^{-6}y^8$$

Step 3: Simplify. Recall that for any non-zero real number b and integer n, $b^{-n} = \frac{1}{b^n}$. Apply this to x^{-6}.

$$(-2x^3y^{-4})^{-2} = \frac{1}{4}x^{-6}y^8$$
$$(-2x^3y^{-4})^{-2} = \frac{y^8}{4x^6}$$

Thus, $(-2x^3y^{-4})^{-2} = \frac{y^8}{4x^6}$.

Try It Yourself!

Simplify $(5x^{-5}y^6)^{-4}$.

Example 3: Simplify the expression $\left(\frac{-4x^4y^{-2}}{5x^{-1}y^4}\right)^{-4}$.

Solution: Apply the Exponent Rules that are applicable.

Step 1: Raise each of the factors inside the parentheses to the exponent -4.

$$\left(\frac{-4x^4y^{-2}}{5x^{-1}y^4}\right)^{-4} = \frac{(-4)^{-4}(x^4)^{-4}(y^{-2})^{-4}}{(5)^{-4}(x^{-1})^{-4}(y^4)^{-4}}$$

Step 2: Apply the Power of a Power Rule.

$$\frac{(-4)^{-4}x^{4(-4)}y^{(-2)(-4)}}{(5)^{-4}x^{(-1)(-4)}y^{4(-4)}} = \frac{(-4)^{-4}(x^{-16})(y^{8})}{(5)^{-4}(x^{4})(y^{-16})}$$

Step 3: Combine like terms and convert all negative exponents to positive exponents. Use the applicable rules.

$$\frac{(-4)^{-4}(x^{-16})(y^{8})}{(5)^{-4}(x^{4})(y^{-16})} = \frac{(5)^{4}(x^{-16})(y^{8})}{(-4)^{4}(x^{4})(y^{-16})}$$

$$\frac{(-4)^{-4}(x^{-16})(y^{8})}{(5)^{-4}(x^{4})(y^{-16})} = \frac{(5)^{4}x^{-16-4}y^{8-(-16)}}{(-4)^{4}}$$

$$\frac{(-4)^{-4}(x^{-16})(y^{8})}{(5)^{-4}(x^{4})(y^{-16})} = \frac{(5)^{4}x^{-20}y^{24}}{(-4)^{4}}$$

$$\frac{(-4)^{-4}(x^{-16})(y^{8})}{(5)^{-4}(x^{4})(y^{-16})} = \frac{625y^{24}}{256x^{20}}$$

Therefore, the given expression can be simplified to $\frac{625y^{24}}{256x^{20}}$.

Try It Yourself!

Simplify the expression $\left(\frac{3x^{-3}y^{4}}{4x^{4}y^{-3}}\right)^{-3}$.

Real-World Problems

Example 4: In the *Warm Up!* activity, how many rabbits are there in habitats A and B after 12 months? How many rabbits are there in all after 12 months?

Solution: After 12 months, the total number of rabbits in Habitat A will be

$$2 \times 3^{4} = 2 \times 81 = 162$$

The total number of rabbits in Habitat B after 12 months will be

$$8 \times 3^4 = 648$$

We can also write the expression in exponential form.

$$2^3 \times 3^4 = 648$$

To find the total population, we can simply add the population of both habitats.

$$162 + 648 = 810$$

We can also use the exponential form.

$$(2 \times 3^4) + (2^3 \times 3^4) = 162 + 648 = 810$$

Therefore, the total population of the rabbits is 810.

Try It Yourself!

Find the population of rabbits in Habitats A and B in Example 4 after 24 months.

Check Your Understanding!

1. Simplify the following expressions .
   a. 100^0
   b. $(25^{-3})^0$
   c. 4^{-2}
   d. $(3^{-3})^2$
   e. $\frac{2^6}{2^8}$
   f. $(4^x)(4^5)$
   g. $(3m^3)(4m^2)$
   h. $\frac{(6x^4)}{3x^4}$
   i. $\frac{4a^2b^3}{2a^5b^2}$
   j. $\frac{16xy^3}{2x^5y^2}$
   k. $\frac{2^3}{3^2} \times \frac{3^4}{2^5}$
   l. $\frac{4^3}{4^4} \times 2^5$
   m. $\frac{3x^2}{y^3} \times \frac{3y^{-2}}{4x^{-3}}$
   n. $(2x^{-3}y^2)^4$
   o. $(7x^4y^{-4}z^{-2})^{-3}$
   p. $\left(\frac{x^4y^{-3}}{5x^{-2}y^5}\right)^{-5}$
   q. $\left(\frac{2x^{-2}y^3}{3x^2y^{-3}}\right)^{-2} \times \frac{(6x^2y^3)^3}{3x^3}$
   r. $\left(\frac{4a^{-3}b^2c^{-3}}{b^4c^{-1}}\right)^3 \times \frac{(-3ab^{-2}c^2)^{-2}}{(a^{-3}b)}$
   s. $\left(\frac{3m^{-3}n^2}{4m^{-2}n^{-1}}\right)^{-3} \times \left(\frac{m^2n^{-2}}{2n^4}\right)^2$
   t. $\left(\frac{-6xy^2}{x^{-2}y}\right)^{-3} \times \left(\frac{3x^{-2}y^2}{x^2y^{-3}}\right)^3$

2. There are 6^4 trees in a private land. In every tree, there are 6^5 leaves. How many pieces of leaves are there in all? Write your answer in exponential form.

Lesson 2: Writing Expressions with Rational Exponents as Radical Expressions

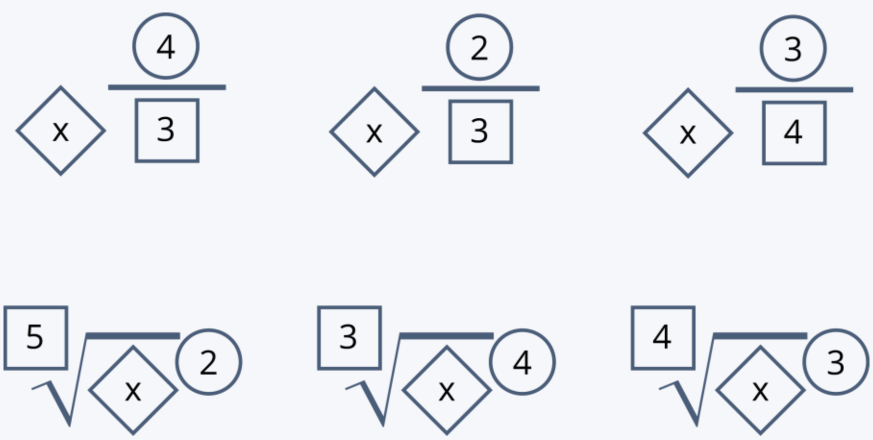
Warm Up!

Follow the Pattern, and Match my Shape!

Materials Needed: pen and paper

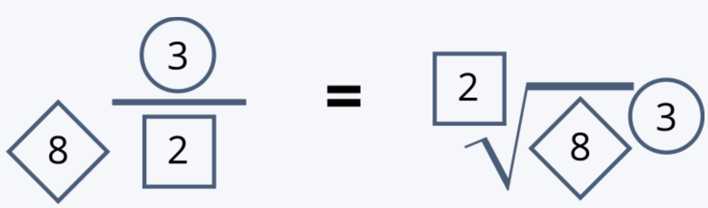
Instructions:

1. This is a whole class activity.
2. Each student will hold a board bearing one of the six figures below.



3. At the count of 5, each student will find his/her partner based on the example below. Lead the students to identify the pattern so that they can look for their correct partner.

PATTERN:



Learn about It!

The *Warm Up!* activity in itself gives us the idea on how to write expressions with rational exponents to radical expressions, and vice versa.

Rational exponents and **radical expressions** are closely linked with each other.

> **Definition 2.1:** A **radical expression** is an expression containing a **radical** ($\sqrt{}$) symbol.

The expression $\sqrt[3]{125}$ is a radical expression, where 3 is called the **index**, $\sqrt{}$ is the **radical sign**, and 125 is called the **radicand**. The index must always be a positive integer greater than 1. If there is no index is written, then it is assumed to be 2, or a square root. If the index is 3, then it is a cube root. Generally, we say nth root for indexes n.

In this lesson, let us take a closer look at how the concepts of **rational exponents** and **radical expressions** are interchangeable.

Let us dissect the relationship with a basic example that we are more familiar with:

$$\sqrt{x}$$

The expression above is read as the **square root of x raised to 1.** Now, this is the part when we convert the radical expression into a rational exponent. We have to be mindful of the **kind of root** and the **degree** of the variable inside the radical.

In this case, we are taking the square **(2^{nd})** root, and the **degree of the variable is 1**.

Given these information, we can convert the example this way:

$$\sqrt{x} = x^{\frac{1}{2}}$$

It is now read as x **raised to one-half**. We can generalize this to the form:

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

The rational exponent is **the degree of the variable divided by the nth-root.**

Let's Practice!

Example 1: Rewrite $\sqrt[3]{x^4}$ as a variable with a rational exponent.

Solution: Determine the nth root and the degree of the variable.

Step 1: Identify the nth root.

$$\sqrt[3]{x^4}$$

In this case, the nth root is 3.

Step 2: Identify the degree of the variable.

$$\sqrt[3]{x^4}$$

In this case, the degree of the variable is 4.

Step 3: Apply the identity of rational exponents, with the exponent being the degree of the variable over the nth-root.

$$\sqrt[3]{x^4} = x^{\frac{4}{3}}$$

Thus, the radical expression is now translated to $\boldsymbol{x^{\frac{4}{3}}}$**, or x raised to four-thirds.**

Try It Yourself!

Rewrite $\sqrt[4]{x^3}$ as a variable with a rational exponent.

Example 2: Rewrite the expression $x^{\frac{4}{5}}$ into a radical expression.

Solution: Identify which part of the radical exponent goes into the root and the variable.

Step 1: The denominator 5 goes into the root.

$$\sqrt[5]{x}$$

Step 2: The numerator 4 is the degree of the variable.

$$\sqrt[5]{x^4}$$

Therefore, the new form of $x^{\frac{4}{5}}$ is $\sqrt[5]{x^4}$.

Try It Yourself!

Rewrite the expression $x^{\frac{3}{4}}$ into a radical expression.

Example 3: Simplify the expression $8^{\frac{2}{3}}$.

Solution: Convert the expression into a radical expression, then perform the operation.

Step 1: The denominator 3 goes into the root.

$$\sqrt[3]{8}$$

Step 2: The numerator 2 is the exponent of the radicand.

$$\sqrt[3]{8^2}$$

Step 3: Simplify 8^2 to get 64.

$$\sqrt[3]{64}$$

Step 4: Solve for the cube root of 64. This yields the value 4.

$$\sqrt[3]{4 \times 4 \times 4} = \sqrt[3]{4^3} = 4$$

Note that if $\sqrt[3]{4^3}$ is converted into rational exponent, we have

$$(4^3)^{\frac{1}{3}} = 4$$

Thus, we can say that the exponent and the root cancel each other.

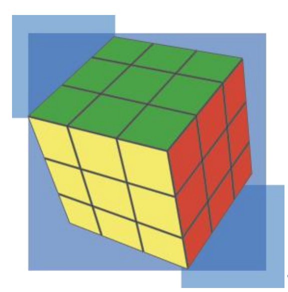
Therefore, the simplified value of $8^{\frac{2}{3}}$ is 4.

Try It Yourself!

Simplify the expression $4^{\frac{3}{2}}$.

Real-World Problems

Example 4: The length of a side of a Rubik's cube is given by $\left(\frac{729}{64}\right)^{\frac{1}{3}}$ inches. Simplify the given expression.



Solution: Convert the expression into a radical expression, then perform the operation.

Step 1: The denominator 3 goes into the root.

$$\sqrt[3]{\frac{729}{64}}$$

Step 2: The numerator 1 is the exponent of the radicand.

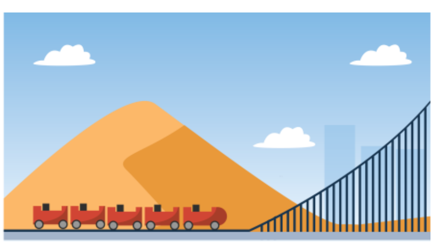
$$\sqrt[3]{\left(\frac{729}{64}\right)^1}$$

Step 3: Simplify $\sqrt[3]{\frac{729}{64}}$. This yields the value $\frac{9}{4}$ since $\left(\frac{9}{4}\right)^3 = \frac{729}{64}$.

$$\sqrt[3]{\frac{729}{64}} = \frac{9}{4}$$

Thus, the length of a side of a Rubik's cube is $\frac{9}{4}$ inches.

Try It Yourself!



The speed s of a roller coaster (in feet per second) at the foot of a hill is approximated as $s = \sqrt{128h}$, where h is the height of the hill in feet. Write the given speed into a rational exponent.

Check Your Understanding!

1. Rewrite the following radical expressions into a rational exponent.
   a. $\sqrt[4]{x^2}$
   b. $\sqrt[3]{a^7}$
   c. $\sqrt[5]{x^4}$
   d. $\sqrt[3]{x^2y^5}$
   e. $\sqrt[4]{m^5n^3}$

2. Rewrite the rational exponents into radical expressions.
   a. $3^{\frac{2}{3}}$
   b. $x^{\frac{3}{8}}$
   c. $(ab)^{\frac{2}{3}}$
   d. $(x^2y^3)^{\frac{3}{4}}$
   e. $(2ab^2)^{\frac{5}{4}}$

3. Simplify the following expressions.
   a. $\sqrt[3]{2^5}$
   b. $\sqrt[4]{3^6}$
   c. $5^{\frac{4}{3}}$
   d. $(x^2y^3)^{\frac{5}{3}}$
   e. $\sqrt[3]{12x^5y^4}$
   f. $36^{\frac{1}{2}}$
   g. $1000000^{\frac{1}{6}}$
   h. $\left(\frac{125}{343}\right)^{\frac{-1}{3}}$
   i. $(-16)^{\frac{-1}{4}}$
   j. $(x^3y^2)^{\frac{1}{2}}$

4. The length of a rectangular lot is given by the expression $\left(\frac{125}{8}\right)^{\frac{1}{3}}$ inches. Simplify the given expression.

Lesson 3: The Laws of Radicals

Warm Up!

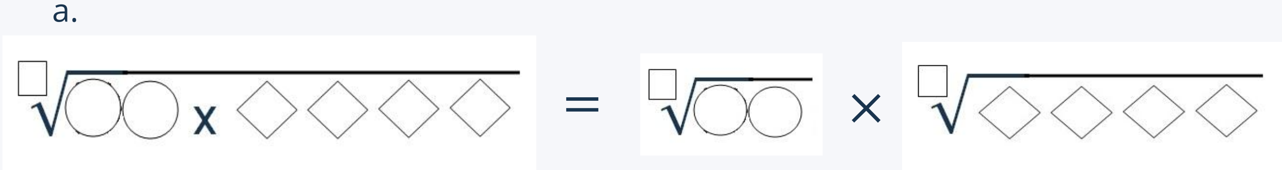
Make Fun and Learn with Shapes!

Materials Needed: pen and paper

Instructions:

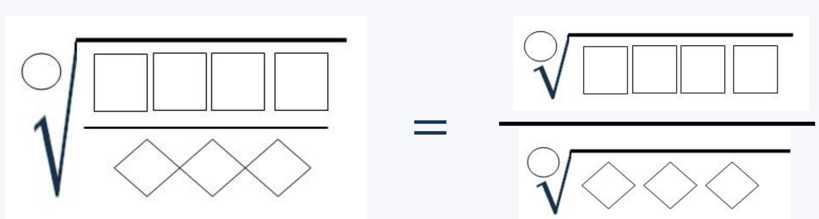
1. This activity can be done individually or race to finish first with a partner.
2. Below are shapes corresponding to numbers.
3. Write the exponents and radicals as described in the figure.

a.



Numerical equivalent: _______________ = ____________ + _____________

b.



Numerical equivalent: _______________ = __________________

Learn about It!

Now that we have learned how to relate exponents to radicals, let us study further the concept of radicals. In this lesson, we will delve deeper into the laws that govern how radical expressions are evaluated.

The *Warm Up!* activity demonstrates the first two rules of radical. Let us relate what you have accomplished with these rules.

The following rules are important points to note when trying to evaluate radicals.

1. **Product Rule for Simplifying Radical Expressions**

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Example: $\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8} \cdot \sqrt[3]{27} = -2 \cdot 3 = -6$

2. **Quotient Rule for Simplifying Radical Expressions**

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Example: $\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{\sqrt[4]{2^4}}{\sqrt[4]{3^4}} = \frac{2}{3}$

3. **The m-th root of an n-th root Rule**

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Example: $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = \sqrt[6]{3^6} = 3$

Let's Practice!

Example 1: Simplify the expression $\sqrt[6]{y^5} \cdot \sqrt[6]{y^2}$, and write the corresponding expression in rational exponents.

Solution:

Step 1: Apply the Product Rule.

$$\sqrt[6]{y^5} \cdot \sqrt[6]{y^2} = \sqrt[6]{y^5 \cdot y^2}$$

Step 2: Apply the rule of multiplying exponents with the same base.

$$\sqrt[6]{y^5 \cdot y^2} = \sqrt[6]{y^7}$$

Step 3: Apply the rule on conversion from radical expressions to rational exponents.

Given that the degree of the variable is 7 and the nth root is 6, we write the rational exponent as:

$$\sqrt[6]{y^7} = y^{\frac{7}{6}}$$

Therefore, the expression is simplified into $y^{\frac{7}{6}}$.

Try It Yourself!

Simplify the expression $(\sqrt[5]{x^4})(\sqrt[5]{x^3})$, and write the corresponding expression in rational exponents.

Example 2: Simplify the expression $\frac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}}$.

Solution:

Step 1: Apply the Quotient Rule.

$$\frac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}} = \sqrt[4]{\frac{x^7}{x^3}}$$

Step 2: Apply the rule of dividing exponents with the same base.

$$\sqrt[4]{\frac{x^7}{x^3}} = \sqrt[4]{x^{7-3}}$$

$$\sqrt[4]{\frac{x^7}{x^3}} = \sqrt[4]{x^4}$$

Step 3: Apply the rule on conversion of radical expressions to rational exponents.

Given that the degree of the variable is 4 and the nth root is 4, we write the rational exponent as:

$$\sqrt[4]{x^4} = x^{\frac{4}{4}} = x$$

Thus, the expression is simplified into x^1, which is simply x.

Try It Yourself!

Simplify the expression $\frac{\sqrt[5]{y^7}}{\sqrt[5]{y^4}}$.

Example 3: Simplify the expression $\sqrt[3]{\sqrt[4]{y}}$, and write the corresponding expression in rational exponents.

Solution:

Step 1: Apply the m-th root of an n-th root Rule.

$$\sqrt[3]{\sqrt[4]{y}} = \sqrt[3\cdot 4]{y} = \sqrt[12]{y}$$

Step 2: Apply the rule on conversion of radical expressions to rational exponents.

Given that the degree of the variable is 1 and the n-th root is 12, we write the rational exponent as:

$$\sqrt[12]{y} = y^{\frac{1}{12}}$$

Thus, the expression is simplified into $y^{\frac{1}{12}}$.

Try It Yourself!

Simplify the expression $\sqrt[5]{\sqrt[3]{x^2}}$, and write the corresponding expression in rational exponents.

Real-World Problems

Example 4: A rectangular lot has a length of $25\sqrt{5}$ meters and a width of $18\sqrt{10}$ meters. What is the area of the lot?

Solution:

Step 1: List the given information.

Length = $25\sqrt{5}$ meters
Width = $18\sqrt{10}$ meters

Step 2: Write the working equation.

$$\text{Area} = \text{length} \times \text{width}$$

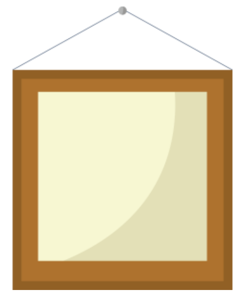
Step 3: Perform the indicated operation.

$A = l \times w$	
$A = (25\sqrt{5})(18\sqrt{10})$	Apply the Product Rule on the expressions $\sqrt{5}$ and $\sqrt{10}$ to obtain $\sqrt{50}$.
$A = 450\sqrt{50}$	Simplify $\sqrt{50}$ by expressing the radicand as a product of its factors, where one of the factors is a perfect square.
$A = 450\sqrt{25 \times 2}$	
$A = 450\sqrt{5^2 \times 2}$	Apply the Product Rule on the expression $\sqrt{5^2 \times 2}$.
$A = 450\sqrt{5^2} \times \sqrt{2}$	
$A = 450 \cdot 5\sqrt{2}$	
$A = 2250\sqrt{2}$	

Therefore, the area of the rectangular lot is $2250\sqrt{2}$ square meters.

Try It Yourself!

The length of a side of a square picture frame is $12\sqrt{3}$ inches. What is the area of the picture frame?



Check Your Understanding!

1. Simplify the following expressions, then write them in rational exponents.
   a. $\sqrt[5]{3^4} \cdot \sqrt[5]{3^6}$
   b. $(\sqrt[4]{2^8})(\sqrt[4]{2^3})$
   c. $(\sqrt[4]{abc})(\sqrt[4]{a^2b^2c^2})$
   d. $(\sqrt[3]{27x^2})(\sqrt[3]{3x^3})$
   e. $(\sqrt[5]{x^4y^3})(\sqrt[5]{x^2y^4})$
   f. $\frac{\sqrt[3]{6^4}}{\sqrt[3]{6^2}}$
   g. $\frac{\sqrt[4]{2^4}}{\sqrt[4]{2^8}}$
   h. $\frac{\sqrt[3]{216x^{12}}}{\sqrt[3]{8x^5}}$
   i. $\frac{\sqrt[4]{64y^5}}{\sqrt[4]{4y^9}}$
   j. $\frac{\sqrt[5]{729x^8}}{\sqrt[5]{3x^3}}$
   k. $\frac{\sqrt[3]{x^2y^4}}{\sqrt[3]{xy}}$
   l. $\frac{\sqrt[6]{8a^5b^7}}{\sqrt[6]{2a^2b}}$
   m. $\sqrt[3]{\sqrt[3]{9}}$
   n. $\sqrt[4]{\sqrt[5]{xy^2}}$
   o. $\sqrt[3]{\sqrt{64}}$
   p. $\sqrt[3]{16x^2} \cdot \frac{\sqrt[3]{3x^5}}{\sqrt[3]{x^4}}$
   q. $\frac{\sqrt[5]{4x^2y^3}}{\sqrt[5]{xy}} \cdot \sqrt[5]{5x^3y^4}$
   r. $\frac{\sqrt[3]{16a^5b^7}}{\sqrt[4]{27x^2y^2}} \cdot \frac{\sqrt[3]{4ab^2}}{\sqrt[4]{3x^2y^2}}$
   s. $\sqrt[5]{m^4n^6} \cdot \frac{\sqrt[5]{m^2n^2}}{\sqrt[5]{n^3}}$
   t. $\frac{\sqrt[3]{8x^4y^5}}{\sqrt[3]{xy^5}} \cdot \frac{\sqrt[4]{24x^3y^2}}{\sqrt[4]{3xy^2}}$

2. The length of a side of a notepad is given by the expression $\sqrt[3]{\frac{3\,375}{64}}$ inches. Simplify the given radical expression.

Lesson 4: Addition and Subtraction of Radicals

Warm Up!

Provide Me the Rule!

Materials Needed: pen and paper

Instructions:

1. This activity should be done individually.
2. Study the information given at the table.
3. Provide the rule based on the information given.

Expression A	Expression B	Sum/Difference
$3\sqrt{2}$	$5\sqrt{2}$	$8\sqrt{2}$
$6\sqrt{5}$	$\sqrt{5}$	$5\sqrt{5}$
$4\sqrt{3}+2\sqrt{6}$	$\sqrt{3}-\sqrt{6}$	$5\sqrt{3}+\sqrt{6}$
$4\sqrt[3]{5}$	$7\sqrt[3]{5}$	$-3\sqrt[3]{5}$
$6\sqrt[4]{3}-3\sqrt[3]{2}$	$4\sqrt[4]{3}-5\sqrt[3]{2}$	$10\sqrt[4]{3}-8\sqrt[3]{2}$

Learn about It!

Similar to numbers, radical expressions can also be added and/or subtracted. One important thing to note is that we cannot combine unlike radical terms. Radicals should be **similar**, which means that they are of **the same index and radicand** in order to be added or subtracted.

Take note that before adding or subtracting radicals, simplify any term with a perfect square radicand.

Example: $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$

After this step has been completed, we can now add or subtract radicals.

You may have noted some observations in the *Warm Up!* activity which made you come up with rules. The first, third, and fifth rows in the table demonstrate the addition of radicals, while the second and the fourth row show the subtraction of radicals.

In adding or subtracting radicals, we add the coefficients, then copy the radical.

Let's Practice!

Example 1: Find the sum of $5\sqrt{27}$ and $\sqrt{48}$.

Solution:

Step 1: Simplify each term.

$$5\sqrt{27} = 5\sqrt{9 \times 3} \qquad \sqrt{48} = \sqrt{16 \times 3}$$
$$5\sqrt{27} = 5\sqrt{3^2 \times 3} \qquad \sqrt{48} = \sqrt{4^2 \times 3}$$
$$5\sqrt{27} = (5)3\sqrt{3} \qquad \sqrt{48} = 4\sqrt{3}$$
$$5\sqrt{27} = 15\sqrt{3}$$

Step 2: Add the coefficients, and copy the radical.

$$15\sqrt{3} + 4\sqrt{3} = (15 + 4)\sqrt{3} = 19\sqrt{3}$$

Thus, $5\sqrt{27} + \sqrt{48} = 19\sqrt{3}$.

Try It Yourself!

Find the sum of $\sqrt{48}$ and $\sqrt{12}$.

Example 2: Find the difference between $4\sqrt{338}$ and $\sqrt{128}$.

Solution:

Step 1: Simplify each term.

$$4\sqrt{338} = 4\sqrt{169 \times 2} \qquad \sqrt{128} = \sqrt{64 \times 2}$$
$$4\sqrt{338} = 4\sqrt{13^2 \times 2} \qquad \sqrt{128} = \sqrt{8^2 \times 2}$$
$$4\sqrt{338} = (4)13\sqrt{2} \qquad \sqrt{128} = 8\sqrt{2}$$
$$4\sqrt{338} = 52\sqrt{2}$$

Step 2: Subtract 8 from 52, then copy the radical.

$$52\sqrt{2} - 8\sqrt{2} = (52 - 8)\sqrt{2} = 44\sqrt{2}$$

Therefore, $4\sqrt{338} - \sqrt{128} = 44\sqrt{2}$.

Try It Yourself!

Find the difference of $2\sqrt{1458} - 5\sqrt{162}$.

Example 3: Simplify the expression: $\sqrt{18} - 2\sqrt{27} + 3\sqrt{3} - 6\sqrt{8}$

Solution:

Step 1: Simplify the terms $\sqrt{18}, 2\sqrt{27}$, and $6\sqrt{8}$. The expression $3\sqrt{3}$ is already simplified.

$\sqrt{18} = \sqrt{9 \times 2}$
$\sqrt{18} = \sqrt{3^2 \times 2}$
$\sqrt{18} = 3\sqrt{2}$

$2\sqrt{27} = 2\sqrt{9 \times 3}$
$2\sqrt{27} = 2\sqrt{3^2 \times 3}$
$2\sqrt{27} = (2)3\sqrt{3}$
$2\sqrt{27} = 6\sqrt{3}$

$6\sqrt{8} = 6\sqrt{4 \times 2}$
$6\sqrt{8} = 6\sqrt{2^2 \times 2}$
$6\sqrt{8} = (6)2\sqrt{2}$
$6\sqrt{8} = 12\sqrt{2}$

The expression can be simplified as $3\sqrt{2} - 6\sqrt{3} + 3\sqrt{3} - 12\sqrt{2}$

Step 2: Combine like terms.

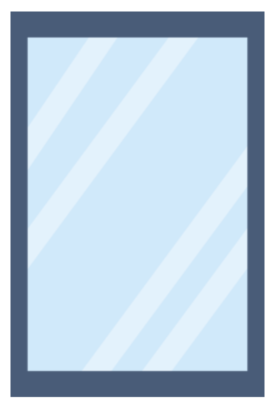
$$3\sqrt{2} - 6\sqrt{3} + 3\sqrt{3} - 12\sqrt{2} = \left(3\sqrt{2} - 12\sqrt{2}\right) + \left(-6\sqrt{3} + 3\sqrt{3}\right)$$
$$3\sqrt{2} - 6\sqrt{3} + 3\sqrt{3} - 12\sqrt{2} = -9\sqrt{2} - 3\sqrt{3}$$

Therefore, $\sqrt{18} - 2\sqrt{27} + 3\sqrt{3} - 6\sqrt{8} = -9\sqrt{2} - 3\sqrt{3}$.

Try It Yourself!

Simplify the expression: $\sqrt{112} + \sqrt{448} - 5\sqrt{28} - 2\sqrt{7}$

Real-World Problems



Example 4: A rectangular mirror has a length of $36\sqrt{6}$ inches and a width of $12\sqrt{6}$ inches. Find the perimeter of the mirror.

Solution:

Step 1: List the given information.

Length = $36\sqrt{6}$ inches
Width = $12\sqrt{6}$ inches

Step 2: Write the working equation.

$$\text{Perimeter} = 2(\text{length} + \text{width})$$

Step 3: Perform the indicated operation.

$$P = 2(36\sqrt{6} + 12\sqrt{6})$$
$$P = 2(48\sqrt{6})$$
$$P = 96\sqrt{6}$$

Therefore, the perimeter of the rectangular mirror is $96\sqrt{6}$ inches.

Try It Yourself!

The length and width of a bedding is $\sqrt{1800}$ inches and $\sqrt{1250}$ inches, respectively. What is the perimeter of the bedding?

Check Your Understanding!

1. Perform the indicated operation, and simplify if necessary.
   a. $3\sqrt{3} + 5\sqrt{3}$
   b. $2\sqrt{5} + 3\sqrt{5} + \sqrt{5}$
   c. $\sqrt{18} + \sqrt{162}$
   d. $3\sqrt{48} + \sqrt{12}$
   e. $4\sqrt{18} + 2\sqrt{162}$
   f. $8\sqrt{2} - 5\sqrt{2}$
   g. $7\sqrt{3} - 15\sqrt{3}$
   h. $4\sqrt{20} - \sqrt{80}$
   i. $\sqrt{72} + 5\sqrt{2}$
   j. $5\sqrt{45} - \sqrt{405}$
   k. $\sqrt{98} + \sqrt{50}$
   l. $3\sqrt{20} - 6\sqrt{125} + 5\sqrt{45}$
   m. $4\sqrt{63} - 2\sqrt{7} + 5\sqrt{63} - \sqrt{567}$
   n. $2\sqrt{48} - \sqrt{12} - 2\sqrt{192} + \sqrt{768}$
   o. $6\sqrt{2} + \sqrt{150} - \sqrt{250} - 10\sqrt{2}$
   p. $\sqrt{18xy^2} - \sqrt{8xy^2} + \sqrt{50xy^2}$
   q. $5\sqrt[3]{3} - 6\sqrt[3]{3} + 10\sqrt[3]{3}$
   r. $-5\sqrt[4]{x^2} + 4\sqrt[4]{x^2} - 3\sqrt[4]{x^2}$
   s. $10\sqrt{x} - \sqrt{4x} + 2\sqrt{16x}$
   t. $3\sqrt{36x} + \sqrt{100x} - 2\sqrt{64x}$

2. A curtain has a length of $35\sqrt{12}$ inches and a width of $12\sqrt{75}$ inches. What is the perimeter of the curtain?

Lesson 5: Multiplication of Radicals

Warm Up!

Decode Me If You Can!

Materials Needed: pen and paper

Instructions:

1. The activity will be done individually.
2. Use your recent knowledge in simplifying radicals to decode the given information.
3. The given factors in the first three columns in the table are multiplied, and the answer is given in the fourth column.
4. Simplify the products to get the corresponding letter.
5. Read the letters in the product downwards to get the correct word.

Factor	Factor	Factor	Product	Letter
$\sqrt{2}$	$\sqrt{6}$	1	$\sqrt{12}$	
$4\sqrt{3}$	$\sqrt{6}$	1	$4\sqrt{18}$	
$\sqrt[3]{2}$	$\sqrt[3]{4}$	1	$\sqrt[3]{8}$	
$\sqrt{3}$	$\sqrt{4}$	$\sqrt{2}$	$\sqrt{24}$	
$\sqrt{8}$	$\sqrt{3}$	$\sqrt[3]{4}$	$\sqrt{24} \cdot \sqrt[3]{4}$	

A – 2 E - $12\sqrt{2}$ L - $2\sqrt{3}$ N - $2\sqrt{6} \cdot \sqrt[3]{4}$ R - $2\sqrt{6}$

Learn about It!

The *Warm Up!* activity showed you a glimpse on multiplication of radicals. Notice that the products should always be simplified.

How do we multiply radical expressions?

Multiplying Radicals with the Same Index

The Product Rule for Radicals states that:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

This rule can be used if the indices of the radical expressions to be multiplied are the same.

Example: What is the product of $\sqrt{2}$ and $\sqrt{6}$?

Solution: Multiply the radicands, keep the same index, then simplify.

$$\sqrt{2} \cdot \sqrt{6} = \sqrt{2 \cdot 6}$$
$$\sqrt{2} \cdot \sqrt{6} = \sqrt{12}$$
$$\sqrt{2} \cdot \sqrt{6} = \sqrt{4 \cdot 3}$$
$$\sqrt{2} \cdot \sqrt{6} = 2\sqrt{3}$$

Note that if there are coefficients, or the numbers outside the radical, we simply multiply the coefficients, and proceed with the rule in multiplying radicals with same indices.

Multiplying Radicals with Different Indices

Example: What is the product of $\sqrt[3]{2}$ and $\sqrt{6}$?

Solution:

Step 1: Write each in exponential notation.

$$\sqrt[3]{2} \cdot \sqrt{6} = 2^{\frac{1}{3}} \cdot 6^{\frac{1}{2}}$$

Step 2: Write the exponents in terms of their LCD.

The LCD of $\frac{1}{3}$ and $\frac{1}{2}$ is 6.

$$\sqrt[3]{2} \cdot \sqrt{6} = 2^{\frac{1}{3}} \cdot 6^{\frac{1}{2}}$$
$$\sqrt[3]{2} \cdot \sqrt{6} = 2^{\frac{2}{6}} \cdot 6^{\frac{3}{6}}$$

Step 3: Express in radical notation.

$$\sqrt[3]{2} \cdot \sqrt{6} = 2^{\frac{2}{6}} \cdot 6^{\frac{3}{6}}$$
$$\sqrt[3]{2} \cdot \sqrt{6} = \sqrt[6]{2^2} \cdot \sqrt[6]{6^3}$$

Step 4: Simplify the radicands, then apply the rule in multiplying radicals with the same index.

$$\sqrt[3]{2} \cdot \sqrt{6} = \sqrt[6]{2^2} \cdot \sqrt[6]{6^3}$$
$$\sqrt[3]{2} \cdot \sqrt{6} = \sqrt[6]{4} \cdot \sqrt[6]{216}$$
$$\sqrt[3]{2} \cdot \sqrt{6} = \sqrt[6]{864}$$

Step 5: Simplify the radical expression.

$$\sqrt[3]{2} \cdot \sqrt{6} = \sqrt[6]{864}$$

The expression $\sqrt[6]{864}$ is already simplified. Thus, the product of $\sqrt[3]{2}$ and $\sqrt{6}$ is $\sqrt[6]{864}$.

Let's Practice!

Example 1: Find the product of $\sqrt{18}$ and $\sqrt{16}$.

Solution:

Step 1: Multiply the numbers inside the radical symbol.

$$\sqrt{18} \cdot \sqrt{16} = \sqrt{18 \cdot 16} = \sqrt{288}$$

Step 2: Simplify.

$$\sqrt{288} = \sqrt{144 \cdot 2} = 12\sqrt{2}$$

Therefore, $\sqrt{18} \cdot \sqrt{16} = 12\sqrt{2}$.

Another way of solving this is to get the factors of the radicand such that one of its factors is a perfect square.

$$\sqrt{18} \cdot \sqrt{16} = \sqrt{3^2 \cdot 2} \cdot \sqrt{4^2}$$
$$\sqrt{18} \cdot \sqrt{16} = 3\sqrt{2} \cdot 4$$
$$\sqrt{18} \cdot \sqrt{16} = 12\sqrt{2}$$

Try It Yourself!

Find the product of $\sqrt{24}$ and $\sqrt{16}$.

Example 2: Multiply: $\sqrt{6} \cdot \sqrt{15} \cdot \sqrt{10}$

Solution:

Step 1: Multiply the numbers inside the radical symbol.

$$\sqrt{6} \cdot \sqrt{15} \cdot \sqrt{10} = \sqrt{6 \cdot 15 \cdot 10} = \sqrt{900}$$

Step 2: Simplify.

$$\sqrt{900} = \sqrt{30 \cdot 30} = \sqrt{30^2} = 30$$

Therefore, $\sqrt{6} \cdot \sqrt{15} \cdot \sqrt{10} = 30.$

Try It Yourself!

Multiply: $(\sqrt{8})(\sqrt{6})(\sqrt{12})$

Example 3: Simplify the expression $(\sqrt[3]{2a})(\sqrt[4]{a})$.

Solution:

Step 1: Write each in exponential notation.

$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = (2a)^{\frac{1}{3}} \cdot a^{\frac{1}{4}}$$

Step 2: Write the exponents in terms of their LCD.

The LCD of $\frac{1}{3}$ and $\frac{1}{4}$ is 12.

$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = (2a)^{\frac{1}{3}} \cdot a^{\frac{1}{4}}$$
$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = (2a)^{\frac{4}{12}} \cdot a^{\frac{3}{12}}$$

Step 3: Express in radical notation.

$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = (2a)^{\frac{4}{12}} \cdot a^{\frac{3}{12}}$$
$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = \sqrt[12]{(2a)^4} \cdot \sqrt[12]{a^3}$$

Step 4: Simplify the radicands, then apply the rule in multiplying radicals with the same index.

$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = \sqrt[12]{(2a)^4} \cdot \sqrt[12]{a^3}$$
$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = \sqrt[12]{16a^4} \cdot \sqrt[12]{a^3}$$
$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = \sqrt[12]{16a^7}$$

Step 5: Simplify the radical expression.

$$\sqrt[3]{2a} \cdot \sqrt[4]{a} = \sqrt[12]{16a^7}$$

The expression $\sqrt[12]{16a^7}$ is already simplified. Thus, the product of $\sqrt[3]{2a}$ and $\sqrt[4]{a}$ is $\sqrt[12]{16a^7}$.

Try It Yourself!

Simplify the expression: $(\sqrt{18x^4y^3})(\sqrt[3]{6x^3y^3})$

Real-World Problems

Example 4: The dimensions of Ricky's aquarium are $10\sqrt{2}$ by $12\sqrt{3}$ by $20\sqrt{3}$ inches. What is the volume of the aquarium?

Solution:

Step 1: List the given information.

Length: $20\sqrt{3}$ inches
Width: $12\sqrt{3}$ inches
Height: $10\sqrt{2}$ inches

Step 2: Write the working equation.

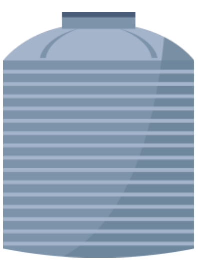
$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$

Step 3: Perform the indicated operation.

$$v = lwh$$
$$v = (20\sqrt{3})(12\sqrt{3})(10\sqrt{2})$$
$$v = (20)(12)(10)(\sqrt{3 \cdot 3 \cdot 2})$$
$$v = 2400\sqrt{18}$$
$$v = 2400\sqrt{9 \cdot 2}$$
$$v = 2400\sqrt{3^2 \cdot 2}$$
$$v = 7200\sqrt{2}$$

Therefore, the volume of the aquarium is $7200\sqrt{2}$ cubic inches.

Try It Yourself!



A cylindrical water tank has a radius of $2\sqrt{3}$ inches and a height of $12\sqrt{3}$ inches. How much water can it hold?

Check Your Understanding!

1. Multiply the following radicals.
   a. $\sqrt{6}\cdot\sqrt{4}$
   b. $\sqrt{8}\cdot\sqrt{12}$
   c. $\sqrt{15}\cdot\sqrt{12}$
   d. $\sqrt{2}\cdot\sqrt{16}$
   e. $\sqrt{18}\cdot\sqrt{9}$
   f. $\sqrt{6}\cdot\sqrt{4}\cdot\sqrt{8}$
   g. $\sqrt{5}\cdot\sqrt{4}\cdot\sqrt{7}$
   h. $\sqrt{6}\cdot\sqrt{12}\cdot\sqrt{8}$
   i. $\sqrt{16}\cdot\sqrt{5}\cdot\sqrt{8}$
   j. $\sqrt{6}\cdot\sqrt{24}\cdot\sqrt{8}$
   k. $\sqrt{16x}\cdot\sqrt{4x^3}$
   l. $\sqrt{2a^3}\cdot\sqrt{3a^4}$
   m. $\sqrt{4x^3y}\cdot\sqrt{6xy^3}$
   n. $\sqrt{9a^5b^3}\cdot\sqrt{8ab^2}$
   o. $\sqrt{10x^3y^5}\cdot\sqrt{12xy}$
   p. $\sqrt[3]{8x^2}\cdot\sqrt[3]{5x^3}$
   q. $\sqrt{12x^3}\cdot\sqrt{3x}$
   r. $\sqrt{x^5y^2}\cdot 5\sqrt{3x^2y^6}$
   s. $2\sqrt[4]{16x^9}\cdot\sqrt[4]{y^3}\cdot\sqrt[4]{81x^{12}y^4}$
   t. $\sqrt[3]{15a^4b^2}\cdot\sqrt[3]{12a^5b^4}$

2. Find the volume of a cube whose side is $8\sqrt{4}$ cm.

Lesson 6: Division of Radicals

Warm Up!

Divide and Observe!

Materials Needed: pen and paper

Instructions:
1. This activity should be done individually.
2. Divide the radicand in Column A by the radicand in Column B, then simplify the answer.

3. Is your answer the same with the expressions in their corresponding cell in Column C? Write YES in the fourth column if you obtained the same answer, and NO if not.

A	B	C	YES or NO?
$\sqrt{18}$	$\sqrt{2}$	3	
$\sqrt[3]{64}$	$\sqrt[3]{4}$	$\sqrt[2]{2}$	
$\sqrt{12x^3}$	$\sqrt{x}$	$2x\sqrt{3}$	
$\sqrt[4]{128a^8}$	$\sqrt[4]{4a^2}$	$2a\sqrt{2a^2}$	
$\sqrt{48}$	$\sqrt{3}$	8	

Learn about It!

The activity in *Warm Up!* somehow gave you a glimpse on how to divide radicals. Let us study the examples below to further understand this concept.

Dividing Radicals with the Same Index

The Quotient Rule for Radicals states that:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

This rule can be used if the indices of the radical expressions to be divided are the same.

Consider the following example: $\frac{\sqrt{24x^3}}{\sqrt{2x}}$.

To simplify it, first, apply the Quotient Rule for radicals.

$$\frac{\sqrt{24x^3}}{\sqrt{2x}} = \sqrt{\frac{24x^3}{2x}}$$
$$\frac{\sqrt{24x^3}}{\sqrt{2x}} = \sqrt{12x^2}$$
$$\frac{\sqrt{24x^3}}{\sqrt{2x}} = \sqrt{4 \cdot 3 \cdot x^2}$$
$$\frac{\sqrt{24x^3}}{\sqrt{2x}} = 2x\sqrt{3}$$

Therefore, $\frac{\sqrt{24x^3}}{\sqrt{2x}} = 2x\sqrt{3}$.

Dividing Radicals with Different Indices

Example: Perform the following division: $\sqrt[3]{a} \div \sqrt[4]{a}$.

Solution:

Step 1: Write the expressions with rational exponents.

$$\sqrt[3]{a} \div \sqrt[4]{a} = \frac{a^{\frac{1}{3}}}{a^{\frac{1}{4}}}$$

Step 2: Apply the Quotient Rule for exponents.

$$\sqrt[3]{a} \div \sqrt[4]{a} = \frac{a^{\frac{1}{3}}}{a^{\frac{1}{4}}}$$
$$\sqrt[3]{a} \div \sqrt[4]{a} = a^{\frac{1}{3}-\frac{1}{4}}$$

Step 3: Simplify the exponents.

$$\sqrt[3]{a} \div \sqrt[4]{a} = a^{\frac{1}{3}-\frac{1}{4}}$$
$$\sqrt[3]{a} \div \sqrt[4]{a} = a^{\frac{1}{12}}$$

Step 4: Write the final answer in radical expression.

$$\sqrt[3]{a} \div \sqrt[4]{a} = a^{\frac{1}{12}}$$
$$\sqrt[3]{a} \div \sqrt[4]{a} = \sqrt[12]{a}$$

If a radical appears at the denominator in the final stage, it must be simplified through **rationalization.**

Example: Simplify $\frac{1}{\sqrt{2}}$.

Solution:

Step 1: Multiply the numerator and the denominator by a radical that will get rid of the radical in the denominator.

Note that multiplying $\sqrt{2}$ in the denominator yields to a radicand that is a perfect square.

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2^2}} = \frac{\sqrt{2}}{2}$$

Step 2: Simplify, if needed.

The expression $\frac{\sqrt{2}}{2}$ is already simplified.

Rationalizing the Denominator with Two Terms

Example: Rationalize the denominator: $\frac{5}{\sqrt{2}+6}$.

Solution:

Step 1: Find the conjugate of the denominator.

Generally, the conjugate of $a+b$ is $a-b$, and vice versa. Thus, the conjugate of $\sqrt{2}+6$ is $\sqrt{2}-6$.

Step 2: Multiply both the numerator and the denominator by the conjugate you obtained in *Step 1*.

$$\frac{5}{\sqrt{2}+6}\cdot\frac{\sqrt{2}-6}{\sqrt{2}-6}=\frac{5\sqrt{2}-30}{\sqrt{4}-6\sqrt{2}+6\sqrt{2}-36}$$
$$\frac{5}{\sqrt{2}+6}\cdot\frac{\sqrt{2}-6}{\sqrt{2}-6}=\frac{5\sqrt{2}-30}{\sqrt{4}-36}$$
$$\frac{5}{\sqrt{2}+6}\cdot\frac{\sqrt{2}-6}{\sqrt{2}-6}=\frac{5\sqrt{2}-30}{2-36}$$
$$\frac{5}{\sqrt{2}+6}\cdot\frac{\sqrt{2}-6}{\sqrt{2}-6}=\frac{5\sqrt{2}-30}{-34}$$

Step 3: Simplify, if needed.

$$\frac{5\sqrt{2}-30}{-34}=-\frac{5\sqrt{2}}{34}+\frac{15}{17}$$

Let's Practice!

Example 1: Simplify $\sqrt{\frac{48}{25}}$.

Solution:

Step 1: Apply the Quotient Rule for simplifying radical expressions.

$$\sqrt{\frac{48}{25}} = \frac{\sqrt{48}}{\sqrt{25}}$$

Step 2: Simplify the numerator and the denominator.

Numerator: $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{4^2 \cdot 3} = 4\sqrt{3}$

Denominator: $\sqrt{25} = 5$

Therefore, $\sqrt{\frac{48}{25}} = \frac{4\sqrt{3}}{5}$.

Try It Yourself!

Simplify $\sqrt{\frac{96}{32}}$.

Example 2: Simplify $\frac{\sqrt[3]{640}}{\sqrt[3]{40}}$.

Solution:

Step 1: Apply the Quotient Rule for radicals.

$$\frac{\sqrt[3]{640}}{\sqrt[3]{40}} = \sqrt[3]{\frac{640}{40}} = \sqrt[3]{16}$$

Step 2: Simplify.

$$\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt{2^3 \cdot 2} = 2\sqrt[3]{2}$$

Therefore, $\frac{\sqrt[3]{640}}{\sqrt[3]{40}} = 2\sqrt[3]{2}$.

Try It Yourself!

Simplify $\frac{\sqrt[3]{512}}{\sqrt[3]{64}}$.

Example 3: Simplify the expression $\sqrt[3]{\frac{54x^2y^4}{2x^4y}}$.

Solution:

Step 1: Simplify the radicand using the laws of exponent.

$$\sqrt[3]{\frac{54x^2y^4}{2x^4y}} = \sqrt[3]{\frac{27x^2y^4}{x^4y}} = \sqrt[3]{\frac{27y^4}{x^2y}} = \sqrt[3]{\frac{27y^3}{x^2}}$$

Step 2: Determine the cube root.

$$\sqrt[3]{\frac{27y^3}{x^2}} = \sqrt[3]{\frac{3^3y^3}{x^2}} = \frac{3y}{\sqrt[3]{x^2}}$$

Step 3: Rationalize the denominator. Note that multiplying $\sqrt[3]{x}$ in the denominator yields to a radicand that is a perfect cube.

$$\frac{3y}{\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{3y\sqrt[3]{x}}{\sqrt[3]{x^3}} = \frac{3y\sqrt[3]{x}}{x}$$

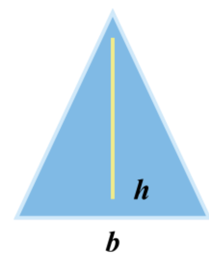
Therefore, the expression $\sqrt[3]{\frac{54x^2y^4}{2x^4y}}$ is simplified into $\frac{3y\sqrt[3]{x}}{x}$.

Try It Yourself!

Simplify $\sqrt[4]{\frac{48x^3y^5}{3x^6y^2}}$.

Real-World Problems

Example 4: The height of a triangle is modelled by $\frac{2\sqrt{3}+5}{3\sqrt{2}}$ cm and its base is $\frac{4\sqrt{2}}{3-\sqrt{3}}$ cm. Write the ratio of the height to the base in its simplest form.



Solution: To write the ratio of the height to the base, we divide $\frac{2\sqrt{3}+5}{3\sqrt{2}}$ by $\frac{4\sqrt{2}}{3-\sqrt{3}}$.

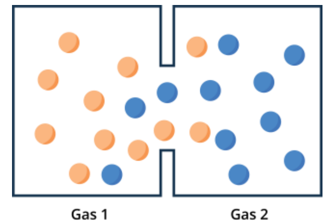
In dividing two fractions, we to multiply the numerator by the reciprocal of the denominator.

$$\frac{2\sqrt{3}+5}{3\sqrt{2}} \cdot \frac{3-\sqrt{3}}{4\sqrt{2}} = \frac{2(3)\sqrt{3}-\left(2\sqrt{3}\right)\left(\sqrt{3}\right)+5(3)-5\sqrt{3}}{12\sqrt{4}}$$
$$\frac{2\sqrt{3}+5}{3\sqrt{2}} \cdot \frac{3-\sqrt{3}}{4\sqrt{2}} = \frac{6\sqrt{3}-2\sqrt{3^2}+15-5\sqrt{3}}{12\sqrt{2^2}}$$
$$\frac{2\sqrt{3}+5}{3\sqrt{2}} \cdot \frac{3-\sqrt{3}}{4\sqrt{2}} = \frac{6\sqrt{3}-2(3)+15-5\sqrt{3}}{12(2)}$$
$$\frac{2\sqrt{3}+5}{3\sqrt{2}} \cdot \frac{3-\sqrt{3}}{4\sqrt{2}} = \frac{6\sqrt{3}-6+15-5\sqrt{3}}{24}$$
$$\frac{2\sqrt{3}+5}{3\sqrt{2}} \cdot \frac{3-\sqrt{3}}{4\sqrt{2}} = \frac{\sqrt{3}+9}{24}$$

Therefore, the ratio of the height to the base of the triangle is $\frac{\sqrt{3}+9}{24}$ cm.

Try It Yourself!

In chemistry, the ratio of the rates of diffusion of two gases is modelled by the formula $\frac{r_1}{r_2} = \frac{\sqrt{m_1}}{\sqrt{m_2}}$ where m_1 and m_2 are masses of the molecules of the gases. Find the ratio of two gases if $m_1 = 16x^8$ and $m_2 = 4x$.



Diffusion

Check Your Understanding!

1. Rationalize the following radical expressions.
   a. $\frac{2}{\sqrt{5}}$
   b. $\frac{3}{\sqrt{3}}$
   c. $\frac{4x}{\sqrt{2}}$
   d. $\frac{5a}{2\sqrt{3a}}$
   e. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}}$

2. Simplify the following radical expressions, and rationalize the denominator, if needed.
   a. $\frac{\sqrt{64}}{\sqrt{2}}$
   b. $\frac{\sqrt{25}}{\sqrt{5}}$
   c. $\frac{6\sqrt{54}}{3\sqrt{2}}$
   d. $\frac{5\sqrt{27}}{\sqrt{3}}$
   e. $\frac{2\sqrt[3]{512}}{\sqrt[3]{64}}$
   f. $2\sqrt{\frac{63}{7}}$
   g. $\sqrt{\frac{24x^3}{3x}}$
   h. $\sqrt{\frac{60x^5y^3}{3x^2y^3}}$
   i. $\sqrt{\frac{144x^7}{36x^5}}$
   j. $\sqrt{\frac{48a^3}{3a^4}}$
   k. $\sqrt{\frac{10m^3n^5}{32m^4n^2}}$
   l. $\frac{\sqrt[3]{375x^{12}y^5}}{\sqrt[3]{25x^3y^2}}$

3. Solve the problem.
   The area of a rectangle is $60\sqrt{3}$ cubic centimetres. Find the length of the rectangle if the length is $5\sqrt{6}$ centimeters.

Lesson 7: Solving Equations Involving Radicals

Warm Up!

Try and Try and Try!

Materials Needed: pen and paper

Instructions:

1. This activity can be done individually or by pair.
2. Using the trial and error method, try to determine the value of x as fast as you can.
   a. $\sqrt{x} = 3$
   b. $\sqrt{x} = 2$
   c. $\sqrt{x+2} = 4$
   d. $\sqrt{x} - 3 = 1$
   e. $\sqrt{x} + 8 = 13$

Learn about It!

Trial-and-error method is a useful way to solve for a value of a variable. However, complicated equations need more strategic computation, and it will take more time to solve when using trial and error.

Now that we have established our skills in evaluating radical expressions, let us now proceed to solving equations with radicals.

Definition 7.1: An equation with a radical expression is called a **radical equation.**

The knowledge about **exponents** and basic **algebraic laws of equality** plays a vital part in studying radical equations.

The basic strategy in solving radical equations is to **isolate the radical term**. Then, **raise both sides of the equation to a power** so that the radical symbol will be removed.

Here are some general rules to note when solving radical equations:

i. If $a = b$, then $a^2 = b^2$. This ensures that in raising both sides of the equation to any power, the result of **both sides will still be equal**.

ii. $\left(\sqrt{x}\right)^2 = x$. We can cancel the radical symbol if the radicand has **same exponent and root**.

Always check if the resulting value satisfies the original equation.

Let's Practice!

Example 1: Solve for the value of x given that $\sqrt{x} = 5$.

Solution:

Step 1: Square both sides of the equation.

$$\left(\sqrt{x}\right)^2 = (5)^2$$
$$x = 25$$

Step 2: Check the result by substituting $x = 25$ to the original equation.

$$\sqrt{x} = 5$$
$$\sqrt{25} = 5$$

The resulting value satisfies the original equation. Thus, $x = 25$.

Try It Yourself!

Solve the equation $\sqrt{x} = 6$.

Example 2: Solve the equation $\sqrt{x} + 2 = 10$.

Solution:

Step 1: Isolate the radical expression by subtracting 2 from both sides of the equation.

$$\sqrt{x} + 2 - 2 = 10 - 2$$
$$\sqrt{x} + 0 = 8$$
$$\sqrt{x} = 8$$

Step 2: Square both sides of the equation to remove the radical symbol.

$$\left(\sqrt{x}\right)^2 = 8^2$$
$$x = 64$$

Step 3: Check the result by substituting $x = 64$ to the original equation.

$$\sqrt{x} + 2 = 10$$
$$\sqrt{64} + 2 = 10$$
$$8 + 2 = 10$$
$$10 = 10$$

The resulting value satisfies the original equation. Thus, $x = 64$.

Try It Yourself!

Solve for the value of x, given that $\sqrt{x} - 9 = -5$.

Example 3: Solve for the value of y given $y = \sqrt{y+7} + 5$.

Solution:

Step 1: Isolate the radical by subtracting 5 from both sides.

$$y - 5 = \sqrt{y+7} + 5 - 5$$
$$y - 5 = \sqrt{y+7} + 0$$
$$y - 5 = \sqrt{y+7}$$

Step 2: Square both sides of the equality to remove the radical symbol.

$$(y-5)^2 = \left(\sqrt{y+7}\right)^2$$
$$(y-5)^2 = y + 7$$

Step 3: Simplify $(y-5)^2$, and combine like terms. Then, equate the resulting expression to 0.

$$(y-5)^2 = y + 7$$
$$(y-5)(y-5) = y + 7$$
$$y^2 - 10y + 25 = y + 7$$
$$y^2 - 10y - y + 25 - 7 = 0$$
$$y^2 - 11y + 18 = 0$$
$$(y-9)(y-2) = 0$$

There are two possible values of y: $y = 2$ and 9.

Step 4: Check if the solution is correct by substituting $y = 2$ and $y = 9$ in the original equation.

Let $y = 2$.

$$y = \sqrt{y+7} + 5$$
$$2 = \sqrt{2+7} + 5$$
$$2 = \sqrt{9} + 5$$
$$2 = 3 + 5$$
$$2 \neq 8$$

Thus, $y = 2$ is not a valid solution.

Let $y = 9$.

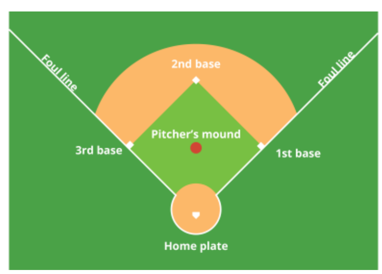
$$y = \sqrt{y+7} + 5$$
$$9 = \sqrt{9+7} + 5$$
$$9 = \sqrt{16} + 5$$
$$9 = 4 + 5$$
$$9 = 9$$

Thus, $y = 9$ is a valid solution to the radical equation $y = \sqrt{y+7} + 5$.

Try It Yourself!

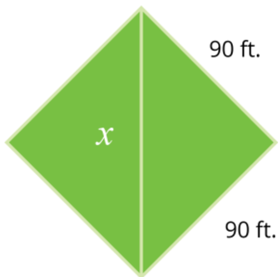
Solve for the value of x given $x = \sqrt{x+7} + 5$.

Real-World Problems



Example 4: A baseball field, which is a square diamond in shape, is 90 feet on each side. How far is the second base from home?

Solution: It is always a good practice to illustrate the problem involving geometric figures.



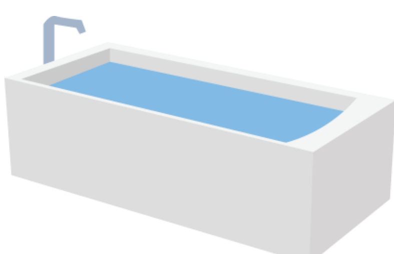
Let x be the distance of the second base from the home base.

Since the field is a square, we can use the Pythagorean Theorem to solve the problem.

$$
\begin{aligned}
x^2 &= a^2 + b^2 \\
x^2 &= 90^2 + 90^2 \\
x^2 &= 2(90^2) \\
\sqrt{x^2} &= \sqrt{2(90^2)} \\
x &= 90\sqrt{2}
\end{aligned}
$$

Therefore, the distance of the second base from the home base is $90\sqrt{2}$ feet.

Try It Yourself!



Find the length of the diagonal of a rectangular bath tub with sides 4 feet and 3 feet.

Check Your Understanding!

1. Solve for the value of x in each item.

   a. $\sqrt{x-5} = 8$

   b. $\sqrt{x+2} = x$

   c. $\sqrt{x+3} = 6$

   d. $\sqrt{x+4} = 1$

   e. $x - 1 = \sqrt{3x-5}$

   f. $6x = 3\sqrt{x} + 4$

   g. $\sqrt{x+3} = x - 9$

   h. $\sqrt{x} + 1 = x - 5$

   o. $\sqrt{3x-1} = 2\sqrt{x-1}$

   i. $\frac{2}{\sqrt{x}} = \frac{4}{x}$

   j. $5 - \sqrt{x+3} = x + 2$

   k. $\frac{1}{\sqrt{x-1}} = \frac{2}{x-1}$

   l. $2\sqrt{x} - 7 = 9$

   m. $3\sqrt{x} + 12 = 9$

   n. $\sqrt{2x+6} = \sqrt{3x+1}$

2. A spherical water tank holds 100 cubic meter of water. Using the formula $d = \sqrt[3]{\frac{6v}{5}}$, find the diameter d of the tank.

Challenge Yourself!

1. One way to measure the amount of energy that a moving object (such as a car) possesses is by finding its kinetic energy. The kinetic energy (E_k, measured in Joules) of an object depends on the object's mass (m, measured in kg) and velocity (v, measured in meters per second), and can be written as $V = \sqrt{\frac{2E}{m}}$. What is the kinetic energy of an object with a mass of 1 000 kilograms that is traveling at 30 meters per second?
2. A young caterpillar may weigh 10^{-2} grams. Is it possible for the caterpillar to grow 10^4 times its body weight during its lifetime?
3. Is it possible to solve for the value of x given $\sqrt[3]{2x-1} = \sqrt[6]{x+1}$?

Performance Task

You are an architect in a construction company. You are tasked to build 3 cylindrical water tanks for a 3-star hotel. You have to design the specifications of the water tanks, specifically the diameters and lengths of the tanks that could hold 1000 m^3, 900 m^3, and 800 m^3 of water. You are required to write a proposal to the company board in two weeks.

Performance Task Rubric

Criteria	Below Expectation (0–49%)	Needs Improvement (50–74%)	Successful Performance (75–99%)	Exemplary Performance (99+%)
Accuracy of Computation and Analysis of Data	There are a significant number of errors in computations that lead to wrong analysis of data.	There are a few errors in the computation but there is no clear basis in the analysis of data.	All computations are correct, and the data are analyzed properly.	All computations are correct and with complete solution. The data are analyzed with clear basis.
Organization of Data/Ideas	Data are not organized properly.	Data are organized properly but some necessary parts are missing.	Data are organized properly. All information needed in the analysis is present.	Data are organized properly. All information needed in the analysis is present.
Promptness of Submission	The proposal is submitted more than 6 days late.	The proposal is submitted 4-6 days late.	The proposal is submitted 1-3 days late.	The proposal is submitted on time.

Wrap-up

Key Concepts & Descriptions

Concept	Description
Product Rule	For any non-zero real number a and integers m and n, $$a^{-m} \cdot a^{-n} = a^{-m+(-n)}$$
Quotient Rule	For any non-zero real number a and integer m and n, $$\frac{a^{-m}}{a^{-n}} = a^{-m-(-n)} = a^{-m+n}$$
Power of a Power Rule	For any non-zero real number a and integer m and n, $$(a^{-m})^{-n} = a^{(-m)(-n)} = a^{mn}$$
Power of a Product Rule	For any non-zero real numbers a and b and integers m, n, p, $$(a^{-m}b^{-n})^{-p} = a^{(-m)(-p)} \times b^{(-n)(-p)}$$
Power of a Quotient Rule	For any non-zero real numbers a and b and integers m, n, p, $$\left(\frac{a^{-m}}{b^{-n}}\right)^{-p} = \frac{a^{(-m)(-p)}}{b^{(-n)(-p)}}$$
Addition and Subtraction of Radicals	Add or subtract terms that have the same root and the same radicand.
Multiplication of Radicals	The Product Rule states that: $$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$
Division of Radicals	The Quotient Rule states that: $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Solving Equations involving Radicals	The basic strategy in solving radical equations is to isolate the radical term. Then, raise both sides of the equation to a power so that the radical symbol will be removed.

Key to *Let's Practice!*

Lesson 1

1. 2187
2. $\frac{x^{20}}{625y^{24}}$
3. $\frac{64x^{21}}{27y^{21}}$
4. 65 610

Lesson 2

1. $x^{\frac{3}{4}}$
2. $\sqrt[4]{x^3}$
3. 8
4. $s = (128h)^{\frac{1}{2}}$

Lesson 3

1. $x^{\frac{7}{5}}$
2. $\sqrt[5]{y^3}$
3. $x^{\frac{2}{15}}$
4. 432 sq. in.

Lesson 4

1. $6\sqrt{3}$
2. $9\sqrt{2}$
3. 0
4. $110\sqrt{2}$ inches

Lesson 5

1. $8\sqrt{6}$
2. 24
3. $6x^3y^3\sqrt{3y}$
4. $144\pi\sqrt{3}$ cubic inches

Lesson 6

1. $\sqrt{3}$
2. 2
3. $\frac{2\sqrt[4]{xy^3}}{x}$
4. $2x^3\sqrt{x}$

Lesson 7

1. 36
2. 16
3. 9
4. 5 feet

References

Bernal, Juvy S., et al. *Conceptual Math & Beyond 9.* Quezon City: Brilliant Creations Publishing, Inc. 2014

MHHE. "Powers and Roots." Accessed May 28, 2018. http://www.mhhe.com/math/devmath/dugopolski/elem/student/olc/graphics/dugopolski03ea_s/ch08/others/ch08-4.pdf